Solutions

Exam 2
Chapter 2.2-2.5 and 3

Name:______________________

Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications. Do not approximate answers. Give exact answers. **Only scientific calculators are allowed on this exam.**

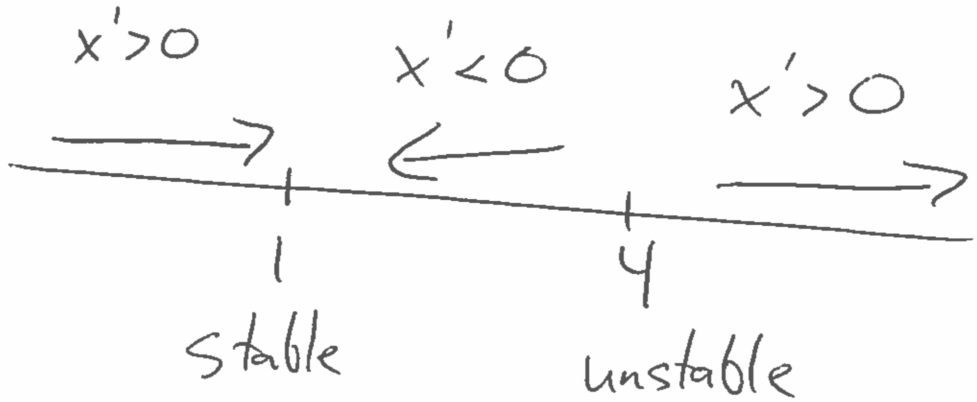
Show your work!

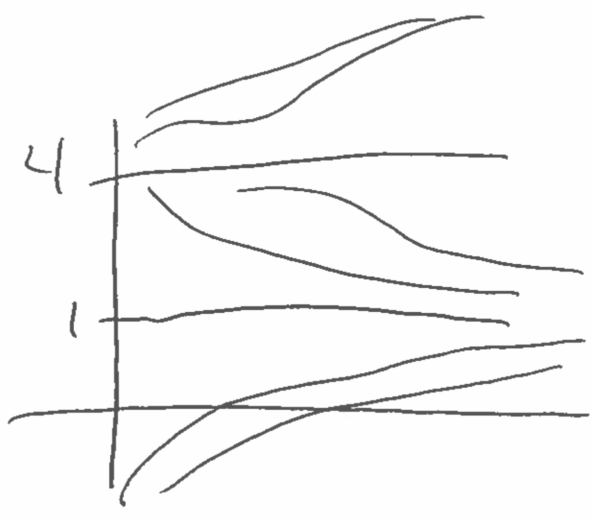
1. (10 points) Draw the phase diagram for the autonomous differential equation

$$\frac{dx}{dt} = x^2 - 5x + 4$$

and determine which critical points are stable and unstable.

$x^2 - 5x + 4 = (x-4)(x-1)$ $\quad x = 1, 4$ are critical pts





2. (10 points) Consider a body that moves horizontally through a medium whose resistance is proportional to the square of velocity so that

$$\frac{dv}{dt} = -2v^2.$$

Assuming that $v(0) = 1$ and $x(0) = 1$, find the position $x(t)$ as a function of t.

$$-\frac{1}{v} = \int \frac{dv}{v^2} = \int -2\,dt = -2t + C$$

$$-1 = -\frac{1}{v(0)} = C$$

Thus $$-\frac{1}{v} = -2t - 1$$

$$\frac{1}{v} = 2t + 1$$

$$v = \frac{1}{2t+1}$$

$$x = \int \frac{1}{2t+1}\,dt = \frac{1}{2}\ln(2t+1) + C$$

$$x(0) = 1 = C$$

So

$$x(t) = 1 + \frac{1}{2}\ln(2t+1).$$

3. (10 points) Find the general form of the complementary solution of the differential equation

$$6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4 = 0$$

which has characteristic function

$$(r^2 + 4)(6r^2 + 5r + 1) = 0.$$

$r_{1,2} = \pm 2i$ $\quad 6r^2 + 5r + 1 = (3r+1)(2r+1)$

$r_3 = \frac{-1}{3}, \; r_4 = \frac{-1}{2}$

$y_c = C_1 e^{-x/3} + C_2 e^{-x/2} + C_3 \cos 2x + C_4 \sin 2x$

4. (10 points) Find the particular solution to the differential equation

$$y'' + 2y' + 2y = 3x^2 - 1.$$

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$3x^2 - 1 = y_p'' + 2y_p' + 2y_p = 2A + 2(2Ax + B) + 2(Ax^2 + Bx + C)$$

$$= 2Ax^2 + (4A + 2B)x + 2A + 2B + 2C$$

$$3 = 2A \Rightarrow A = \frac{3}{2}$$

$$0 = 4A + 2B \Rightarrow 6 + 2B \Rightarrow B = -3$$

$$-1 = 2A + 2B + 2C = 3 - 6 + 2C \Rightarrow C = 1$$

Thus

$$y_p = \frac{3}{2}x^2 - 3x + 1.$$

5. (10 points) Consider an RLC circuit with $R = 30$ ohms, $L = 10$ henries, $C = 0.02$ farads and $E(t) = 50 \sin 2t$ volts at time t. This information gives the differential equation

$$10I'' + 30I' + 50I = 100 \cos 2t.$$

Find the general complementary solution and the particular solution for this circuit.

$$I'' + 3I' + 5I = 10 \cos 2t$$

$$r^2 + 3r + 5 = 0$$

$$r_{1,2} = \frac{-3 \pm \sqrt{9-20}}{2} = \frac{-3 \pm i\sqrt{11}}{2}$$

So that $y_c = e^{-3x/2}\left(c_1 \cos \frac{\sqrt{11}}{2} t + c_2 \sin \frac{\sqrt{11}}{2} t\right)$

$$y_p = A \cos 2t + B \sin 2t$$

$$y_p' = -2A \sin 2t + 2B \cos 2t$$

$$y_p'' = -4A \cos 2t \cancel{+} - 4B \sin 2t$$

$$10 \cos 2t = y_p'' + 3y_p' + 5y_p = -4A \cos 2t - 4B \sin 2t$$
$$+ 3(-2A \sin 2t + 2B \cos 2t)$$
$$+ 5(A \cos 2t + B \sin 2t)$$

$$= (A + 6B) \cos 2t + (B - 6A) \sin 2t.$$

$$10 = A + 6B = A + 36A \Rightarrow A = \frac{10}{37}$$

$$0 = B - 6A \Rightarrow B = 6A \qquad B = \frac{60}{37}$$

$$I(t) = e^{-3x/2}\left(c_1 \cos \frac{\sqrt{11}}{2} t + c_2 \sin \frac{\sqrt{11}}{2} t\right) + \frac{10}{37} \cos 2t + \frac{60}{37} \sin 2t.$$